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Mean-field generation of the classical q-deformation of su (3)

João da Providência
Departamento de Física, Universidade de Coimbra, P3000 Coimbra, Portugal

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Abstract. We investigate conditions under which the classical q-deformation of $su(3)$ is generated by the expectation values of the elements of $su_q(3)$, in q-coherent states. We also discuss the Holstein–Primakoff-type realization of the $su_q(3)$ quantum group in terms of q-bosons.

1. Introduction

Quantum algebras have been attracting much attention in physics. These interesting mathematical structures are generalizations of Lie algebras in which a quantum Yang–Baxter equation replaces, in the associative condition, the usual Jacobi identity. They play a relevant role in a great variety of areas ranging from the theory of integrable systems in statistical physics and field theory to solvable models of molecular and nuclear structure. In the literature, one finds many examples of useful soluble models based on algebraic symmetries. A classic example is the Lipkin model [1], with the $su(2)$ symmetry. Extended Lipkin-type models based on the $su(3)$ symmetry, have also been considered by several authors [2]. The $su(3)$-type models may be described as three-level models associated with the irrep which is fixed by the condition that, in the highest-weight state, all the particles fill up one of the levels. Very often, these models possess a q-deformed counterpart which is interesting to study. In particular, the quantum algebra $su_q(2)$ has been used with success in the description of ground-state bands of some deformed nuclei [3]. Recently, the q-analogue of the Holstein–Primakoff boson realization of $su(2)$ has been obtained by Quesne [4]. A Schwinger-type realization of the $su_q(3)$ algebra in terms of q-boson operators has been given by the same author [5]. Holstein–Primakoff boson realizations of $su(N)$ play a relevant role in connection with the theory of collective states of many-body systems [6].

In this paper we discuss the Holstein–Primakoff-type realization of the $su_q(3)$ quantum group in terms of q-bosons, both at the quantal and at the classical level. The present result extends, therefore, a result of [4] for $su_q(2)$. However, we do not follow, in our discussion, the elegant method of [4] but, instead, we base it on a dynamical formulation in the framework of a mean-field description. The realization of quantum groups in q-deformed oscillator systems, at classical and quantum levels, has been already discussed in detail by Chang *et al.* These authors arrive at q-deformed quantal systems after canonical quantization of the corresponding q-deformed classical system [7]. Our approach is complementary to theirs, in the sense that in our case the q-deformed classical system arises in connection with the mean-field description of the corresponding q-deformed quantal system [8]. We investigate conditions under which the classical q-deformation of $su(3)$ is generated by the expectation values of the elements of $su_q(3)$, in q-coherent states.

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2. Mean-field approach

Following [5], we denote the generators of $su_q(3)$ by E_{ij}, $E_{ii} - E_{jj}$, $i \neq j$, $i, j \in \{1,2,3\}$. Let $|\Phi\rangle$ be such that $E_{23}|\Phi\rangle = E_{21}|\Phi\rangle = E_{13}|\Phi\rangle = E_{31}|\Phi\rangle = 0$, $(E_{22} - E_{11})|\Phi\rangle = (E_{22} - E_{33})|\Phi\rangle = 2j|\Phi\rangle$. In the state $|\Phi\rangle$, regarded as a many-fermion state, the levels 1 and 3 are empty and the level 2 contains $2j$ fermions. The generators $E_{ii} - E_{jj}$ are Hermitian and $E_{23} = E_{32}^{\dagger}$, $E_{21} = E_{12}^{\dagger}$. On the other hand, $E_{13} = E_{12}E_{23} - e^{\gamma}E_{23}E_{12}$ is not the adjoint operator of $E_{31} = E_{32}E_{21} - e^{-\gamma}E_{21}E_{32}$. For details, see [5].

Consider the Fock space spanned by the kets $|mn\rangle = E_{32}^{m}E_{12}^{n}|\Phi\rangle$. The norm of $|mn\rangle$ is

$$\langle mn|mn\rangle = \frac{[2j]![m]![n]!}{[2j-m-n]!}$$

where $[x] = \mathrm{sh}\gamma x/\mathrm{sh}\gamma$. The q-deformed coherent state may be written

$$|\psi(z_1, z_2)\rangle = \sum_{mn} \frac{z_1^m z_2^n}{[m]![n]!}|mn\rangle .$$

The norm of the state $|\psi\rangle = |\psi(z_1, z_2)\rangle$ is

$$\langle\psi|\psi\rangle = \sum_{mn} \frac{(z_1 z_1^*)^m (z_2 z_2^*)^n [2j]!}{[m]![n]![2j-m-n]!} .$$

We may use the q-deformed coherent state $|\psi\rangle$ to describe the dynamics governed by Hamiltonians such as

$$H = E_{33} - E_{22} + g(E_{12}^2 + E_{21}^2 + E_{32}^2 + E_{23}^2 + E_{13}^2 + E_{13}^{\dagger 2}) .$$

This is a q-deformed version of the system considered in [2].

It is convenient to introduce the operators $\hat{N}_1$ and $\hat{N}_2$ defined by

$$\hat{N}_1|m,n\rangle = m|m,n\rangle \qquad \hat{N}_2|m,n\rangle = n|m,n\rangle .$$

In order to write down the quantal action principle we need the following quantity:

$$\frac{\langle\psi|\dot{\psi}\rangle - \langle\dot{\psi}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{z_1^*\dot{z}_1 - \dot{z}_1^* z_1}{z_1^* z_1}\langle\hat{N}_1\rangle + \frac{z_2^*\dot{z}_2 - \dot{z}_2^* z_2}{z_2^* z_2}\langle\hat{N}_2\rangle$$

where $|\dot{\psi}\rangle$ denotes the time derivative of $|\psi\rangle$, and

$$\langle\hat{N}_1\rangle = \langle\psi|\psi\rangle^{-1}\sum_{mn} m\frac{(z_1 z_1^*)^m (z_2 z_2^*)^n [2j]!}{[m]![n]![2j-m-n]!}$$

$$\langle\hat{N}_2\rangle = \langle\psi|\psi\rangle^{-1}\sum_{mn} n\frac{(z_1 z_1^*)^m (z_2 z_2^*)^n [2j]!}{[m]![n]![2j-m-n]!} .$$

In the sequel, the coherent state expectation value of the operator X will be denoted $\langle X\rangle = \langle\psi|X|\psi\rangle/\langle\psi|\psi\rangle$. We also have

$$\frac{\langle[\hat{N}_1]\rangle}{z_1^* z_1} = \frac{\langle[\hat{N}_2]\rangle}{z_2^* z_2} = \langle[2j - \hat{N}_1 - \hat{N}_2]\rangle$$

where

$$\langle[\hat{N}_1]\rangle = \langle\psi|\psi\rangle^{-1}\sum_{mn}[m]\frac{(z_1z_1^*)^m(z_2z_2^*)^n[2j]!}{[m]![n]![2j-m-n]!}$$

$$\langle[\hat{N}_2]\rangle = \langle\psi|\psi\rangle^{-1}\sum_{mn}[n]\frac{(z_1z_1^*)^m(z_2z_2^*)^n[2j]!}{[m]![n]![2j-m-n]!}$$

$$\langle[\hat{2}j-\hat{N}_1-N_2]\rangle = \langle\psi|\psi\rangle^{-1}\sum_{mn}[2j-m-n]\frac{(z_1z_1^*)^m(z_2z_2^*)^n[2j]!}{[m]![n]![2j-m-n]!}\,.$$

Finally, we may write

$$\frac{\langle\psi|\dot{\psi}\rangle-\langle\dot{\psi}|\psi\rangle}{\langle\psi|\psi\rangle} = \left((z_1^*\dot{z}_1-\dot{z}_1^*z_1)\frac{\langle\hat{N}_1\rangle}{\langle[\hat{N}_1]\rangle}+(z_2^*\dot{z}_2-\dot{z}_2^*z_2)\frac{\langle\hat{N}_2\rangle}{\langle[\hat{N}_2]\rangle}\right)\langle[2j-\hat{N}_1-\hat{N}_2]\rangle\,.$$

The following expressions are also easily derived:

$$\begin{aligned}
\langle E_{32}\rangle &= z_1^*\langle[2j-\hat{N}_1-\hat{N}_2]\rangle\\
\langle E_{12}\rangle &= z_2^*\langle[2j-\hat{N}_1-\hat{N}_2]\rangle\\
\langle E_{23}\rangle &= \langle E_{32}\rangle^*\\
\langle E_{21}\rangle &= \langle E_{12}\rangle^*\\
\langle E_{22}-E_{33}\rangle &= 2j-2\langle\hat{N}_1\rangle-\langle\hat{N}_2\rangle\\
\langle E_{13}\rangle &= z_1z_2^*\langle[2j-\hat{N}_1-\hat{N}_2]e^{-\gamma(2j-\hat{N}_1-\hat{N}_2-1)}\rangle\,.
\end{aligned}$$

3. Holstein–Primakoff realization of $su_q(3)$

The variables z_1, z_2, z_1^*, z_2^* obey complicated Poisson bracket (PB) relations. We define new q-oscillator classical variables by

$$\beta_1 = z_1\sqrt{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}\qquad \beta_2 = z_2\sqrt{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}\,.$$

In terms of the classical q-oscillators we may write

$$\frac{\langle\psi|\dot{\psi}\rangle-\langle\dot{\psi}|\psi\rangle}{\langle\psi|\psi\rangle} = (\beta_1^*\dot{\beta}_1-\dot{\beta}_1^*\beta_1)\frac{\langle\hat{N}_1\rangle}{\langle[\hat{N}_1]\rangle}+(\beta_2^*\dot{\beta}_2-\dot{\beta}_2^*\beta_2)\frac{\langle\hat{N}_2\rangle}{\langle[\hat{N}_2]\rangle}\,.$$

We also have

$$\langle E_{32}\rangle = \beta_1^*\sqrt{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}\qquad \langle E_{12}\rangle = \beta_2^*\sqrt{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}$$

$$\langle E_{22}-E_{33}\rangle = 2j-2\langle\hat{N}_1\rangle-\langle\hat{N}_2\rangle\qquad \langle E_{22}-E_{11}\rangle = 2j-\langle\hat{N}_1\rangle-2\langle\hat{N}_2\rangle$$

$$\langle E_{13}\rangle = \beta_1\beta_2^*\frac{\langle[2j-\hat{N}_1-\hat{N}_2]e^{-\gamma(2j-\hat{N}_1-\hat{N}_2-1)}\rangle}{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}\,.$$

$$\langle E_{31}\rangle = \beta_1^*\beta_2\frac{\langle[2j-\hat{N}_1-\hat{N}_2]e^{\gamma(2j-\hat{N}_1-\hat{N}_2-1)}\rangle}{\langle[2j-\hat{N}_1-\hat{N}_2]\rangle}\,.$$

The derivation of $\langle H\rangle$ requires the computation of $\langle E_{ij}^2\rangle$, $i\neq j$. These quantities are also easily obtained. For instance,

$$\langle E_{32}^2\rangle=\beta_1^{*2}\frac{\langle[2j-\hat N_1-\hat N_2][2j-\hat N_1-\hat N_2-1]\rangle}{\langle[2j-\hat N_1-\hat N_2]\rangle}.$$

However, we will focus on $\langle E_{ij}\rangle$.

Although $\langle N_1\rangle$ and $\langle N_2\rangle$ are functions of $\beta_1^*\beta_1$ and $\beta_2^*\beta_2$, it may be found numerically that, to an excellent approximation, $\langle N_1\rangle$ does not depend on $\beta_2^*\beta_2$ and $\langle N_2\rangle$ does not depend on $\beta_1^*\beta_1$. Therefore, the PB relations of the variables β_k^*, β_k may be written [8]

$$\{\beta_k,\beta_l\}=0\qquad\{\beta_k^*,\beta_l^*\}=0\qquad\{\beta_k,\beta_l^*\}=-\mathrm{i}\delta_{kl}\frac{\mathrm{d}[\mathcal{N}_k]}{\mathrm{d}\mathcal{N}_k}$$
$$\{\mathcal{N}_k,\beta_l^*\}=-\mathrm{i}\delta_{kl}\beta_l^*\qquad\{\mathcal{N}_k,\beta_l\}=\mathrm{i}\delta_{kl}\beta_l$$

where $\mathcal{N}_k=\langle\hat N_k\rangle$, $[\mathcal{N}_k]=\beta_k\beta_k^*$. This suggests the following realization of $su_q(3)$ in terms of q-deformed classical oscilators:

$$\mathcal{E}_{32}=\beta_1^*\sqrt{[2j-\mathcal{N}_1-\mathcal{N}_2]}\qquad\mathcal{E}_{23}=\mathcal{E}_{32}^*$$
$$\mathcal{E}_{12}=\beta_2^*\sqrt{[2j-\mathcal{N}_1-\mathcal{N}_2]}\qquad\mathcal{E}_{21}=\mathcal{E}_{12}^*$$
$$\mathcal{E}_{22}-\mathcal{E}_{11}=2j-\mathcal{N}_1-2\mathcal{N}_2\qquad\mathcal{E}_{22}-\mathcal{E}_{33}=2j-2\mathcal{N}_1-\mathcal{N}_2$$
$$\mathcal{E}_{13}=\beta_2^*\beta_1\mathrm{e}^{nnnm-\gamma(2j-\mathcal{N}_1-\mathcal{N}_2-1)}\qquad\mathcal{E}_{31}=\beta_1^*\beta_2\mathrm{e}^{\gamma(2j-\mathcal{N}_1-\mathcal{N}_2-1)}.$$

Typical PB relations satisfied by the generators $\mathcal{E}_{ij}$ are as follows:

$$\{\mathcal{E}_{23},\mathcal{E}_{32}\}=-\mathrm{i}\frac{\gamma}{\mathrm{sh}\gamma}[\mathcal{E}_{22}-\mathcal{E}_{33}]\qquad\{\mathcal{E}_{13},\mathcal{E}_{31}\}=-\mathrm{i}\frac{\gamma}{\mathrm{sh}\gamma}[\mathcal{E}_{11}-\mathcal{E}_{33}]$$
$$\{\mathcal{E}_{23},(\mathcal{E}_{22}-\mathcal{E}_{33})\}=2\mathrm{i}\mathcal{E}_{23}\qquad\{\mathcal{E}_{23},(\mathcal{E}_{22}-\mathcal{E}_{11})\}=\mathrm{i}\mathcal{E}_{23}\qquad\{\mathcal{E}_{12},\mathcal{E}_{32}\}=0$$
$$\{\mathcal{E}_{12},\mathcal{E}_{23}\}=-\mathrm{i}\mathcal{E}_{13}\mathrm{e}^{\gamma(2j-\mathcal{N}_1-\mathcal{N}_2-1)}[2j-\mathcal{N}_1-\mathcal{N}_2]',\ldots$$

where $[x]'=\gamma\mathrm{ch}\gamma x/\mathrm{sh}\gamma$. Notice the factor $\gamma/\mathrm{sh}\gamma$ in the first relations.

The quantal version of the q-deformed oscilators β_i^*, β_i are q-deformed boson operators $b_i^\dagger$, b_i satisfying the q-oscilator algebra

$$b_i^\dagger b_i=[N_i]\qquad b_ib_i^\dagger=[N_i+1]\qquad[b_i^\dagger,b_j]=0,\ i\neq j$$
$$[N_i,b_j^\dagger]=b_i^\dagger\delta_{ij}\qquad[N_i,b_j]=-b_i\delta_{ij}.$$

This suggests the following boson realization of $su_q(3)$:

$$E_{32}^B=b_1^\dagger\sqrt{[2j-N_1-N_2]}\qquad E_{23}^B=E_{32}^{B\dagger}\qquad E_{12}^B=b_2^\dagger\sqrt{[2j-N_1-N_2]}$$
$$E_{21}^B=E_{12}^{B\dagger}\qquad E_{22}^B-E_{11}^B=2j-N_1-2N_2\qquad E_{22}^B-E_{33}^B=2j-2N_1-N_2$$
$$E_{13}^B=b_2^\dagger\mathrm{e}^{-\gamma(2j-N_1-N_2-1)}b_1\qquad E_{31}^B=b_1^\dagger\mathrm{e}^{\gamma(2j-N_1-N_2-1)}b_2.$$

Typical commutation relations satisfied by the generators E_{ij}^B are as follows:

$$[E_{23}^B,E_{32}^B]=[E_{22}^B-E_{33}^B]\qquad[E_{13}^B,E_{31}^B]=[E_{11}^B-E_{33}^B]$$
$$[E_{23}^B,(E_{22}^B-E_{33}^B)]=-2E_{23}^B\qquad[E_{23}^B,(E_{22}^B-E_{11}^B)]=-E_{23}^B$$
$$E_{12}^BE_{32}^B-E_{32}^BE_{12}^B=0\qquad E_{12}^BE_{23}^B-\mathrm{e}^\gamma E_{23}^BE_{12}^B=E_{13}^B,\ldots.$$

The final result may be regarded as an extension, for a particular irrep of $su_q(3)$, of the analogue of the Holstein–Primakoff realization of $su_q(2)$ obtained in [5].

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